Science: A Level 3 Module
SM358 The Quantum World
SM358 AYRF

The Open University

Are you ready for SM358?

Contents

If, after working through these notes, you are still unsure about whether SM358 *The Quantum World* is the right module for you, we advise you to seek further help and advice from your Student Support Team.

1 Introduction

If you are intending to study SM358, you should make sure that you have the necessary background knowledge and skills to be able to enjoy the module fully and to give yourself the best possible chance of completing it successfully.

Read through these notes and work through all the questions in Sections 3 and 4. This is a useful exercise for all prospective students of the module, even those who have studied other Open University science and mathematics modules and who have followed the recommended prior study routes for SM358 (see Section 2 below). Working through these notes and questions will serve as a reminder of some of the knowledge and skills that you are assumed to have, either from OU Level 2 science and mathematics modules or from other prior study or experience.

If you find that you can work through nearly all of the 29 questions in this document within a total of three hours, with only occasional reference to prerequisite material, it is likely that you are well-prepared to start SM358. If you have difficulties with some questions, or take much longer than three hours, this will indicate that you have gaps in your knowledge or that you will need to improve your mathematical fluency.

Section 6 gives advice on specific remedial actions you can take. If you have substantial difficulties with five or more questions, you probably have a considerable amount of catching-up to do, and should seriously assess whether SM358 is the right module to attempt *at this stage of your studies*. Our experience is that students with insufficient preparation find Level 3 physics modules difficult, and often drop out.

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4.1 PT/TEX

2 Suggested prior study

SM358 is a Level 3 module which makes intellectual demands appropriate to the third year of a degree. Quantum physics involves the use of advanced mathematical techniques. All of these techniques are introduced in the prerequisite mathematics modules — either MST224 *Mathematical Methods* or MST210 *Mathematical methods, models and modelling.* It is strongly recommended that you have a *good pass* (grade 1–3) in MST224 or MST210. You should certainly be familiar with complex numbers, vectors, matrices, calculus, differential equations, partial differentiation, polar and spherical coordinates and multiple integrals. It will also be an advantage to understand the basic concepts of Fourier series and partial differential equations at the level of their treatment in MST224 or MST210.

SM358 also assumes that you have previously studied physics, especially mechanics and waves, at an introductory level. A good pass in S217 *Physics: From classical to quantum*, is therefore recommended.

3 Key concepts in physics

To understand the origins of quantum mechanics and its relationship to classical mechanics, previous study of physics at the level of S217 is highly desirable. An understanding of classical mechanics is assumed at the level of S217 Units 1–10. Topics include: Newton's laws, work, kinetic and potential energy, momentum and angular momentum, the force–potential energy relationship and the conservation laws for energy, momentum and angular momentum. This material is also covered by MST210.

Basic knowledge about travelling and standing plane waves is assumed at the level of S217 Unit 14, 17 and 18. Topics include: interference and diffraction, two-slit interference patterns, wavelength, wavenumber, frequency, angular frequency, period, wave speed, amplitude, intensity, nodes and antinodes.

S217 Units 23–26 give a first introduction to quantum physics. However, all the quantum-mechanical concepts introduced in S217 are taught from scratch and in more depth in SM358, so while of S217 is useful background material, it is not essential reading. If you have not studied S217 it is more important to ensure that you have a good grasp of classical mechanics and the physics of waves than to study any prior quantum mechanics.

The following 10 questions test your understanding of basic concepts in physics. You should be able to complete these questions in about an hour. You will need an electronic calculator to answer some of the questions and may find the following data useful:

The elementary charge is $e = 1.60 \times 10^{-19}\,\mathrm{C}$

The electron mass is $m_\mathrm{e} = 9.11 \times 10^{-31}\,\mathrm{kg}$

The constant in Coulomb's law is

$$(4\pi\varepsilon_0)^{-1} = 8.99 \times 10^9\,\mathrm{N\,m^2\,C^{-2}}$$

The speed of light is $c = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$

A vector $\mathbf{v}$ with Cartesian components v_x, v_y and v_z will be denoted by

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are unit vectors pointing in the directions of the x-, y- and z-axes, respectively, of a right-handed Cartesian coordinate system.

Q1 (*momentum and kinetic energy*) Determine

(a) the magnitude of the momentum and

(b) the kinetic energy

of a particle of mass $3.0\,\mathrm{mg}$ moving at a speed of $5.0\,\mathrm{cm\,s^{-1}}$.

Q2 (*momentum in three dimensions*) Write down an expression for the kinetic energy of a particle of mass m with momentum $\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}$.

Q3 (*force and potential energy*) A particle moves along the x-axis in a region where its potential energy function is $V(x) = Cx^4$, where $C = 5.0\,\mathrm{J\,m^{-4}}$ is a constant. Determine the force acting on the particle when it is at $x = 2.0\,\mathrm{m}$.

Q4 (*conservation of energy*)

Figure 1 shows the potential energy function $V(x)$ of a particle on the x-axis. The total energy of the particle (i.e. kinetic plus potential) is $-2\,\mathrm{J}$.

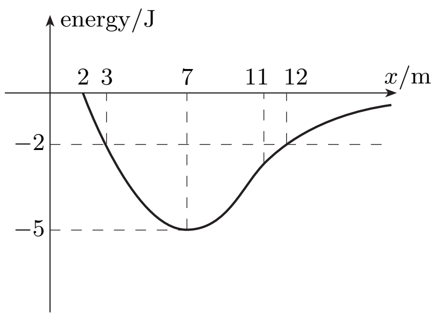

Figure 1 Potential energy function $V(x)$ for Q4.

(a) Calculate the kinetic energy of the particle when it is at $x = 7\,\mathrm{m}$.

(b) Calculate its kinetic energy when it is at $x = 12\,\mathrm{m}$.

(c) What interval of the x-axis is accessible to the particle according to classical physics?

(d) In what direction is the force acting when the particle is at $x = 11\,\text{m}$?

Q5 (*angular momentum*) (a) A particle of mass m moves with velocity $\mathbf{v}$. If its position vector relative to an origin O is $\mathbf{r}$, write down an expression for the angular momentum vector $\mathbf{L}$ of the particle about O in terms of m, $\mathbf{v}$ and $\mathbf{r}$.

(b) What is the angular momentum of the particle relative to the origin when the particle is moving directly towards the origin?

(c) A particle of mass $m = 1.5\,\text{kg}$ has position vector $\mathbf{r} = (2.0\,\text{m})\,\mathbf{j}$ and velocity vector $\mathbf{v} = (3.0\,\text{m}\,\text{s}^{-1})\,\mathbf{i}$. What is the angular momentum of this particle relative to the origin of coordinates?

Q6 (*Hooke's law for a perfect spring*) A particle of mass m at the end of a perfect spring aligned along the x-axis is in equilibrium when it is at position $x = 0$. The force acting on it when it is at position x is $-kx$ where k is the spring constant.

(a) Write down the equation of motion of the particle.

(b) Write down the angular frequency ω of the motion of the particle in terms of its mass m and the spring constant k.

(c) What is the total energy of the particle when it is at position x and is moving with velocity v_x?

Q7 (*Coulomb force law and electrostatic potential energy*) Two stationary protons are separated by $1.00 \times 10^{-11}\,\text{m}$. Determine

(a) the magnitude of the electric force between them;

(b) their electrostatic potential energy (relative to a zero of potential at infinite separation).

Q8 (*standing and travelling waves*) (a) A string of length L is fixed at both ends and vibrates transverse to its own length. Write down a condition that determines the wavelengths of possible standing waves on this string.

(b) A travelling wave takes the form $f(x,t) = A\sin(kx + \omega t)$, where A, k and ω are positive constants. What is the speed of this wave? Does it travel in the positive or negative x-direction?

Q9 (*electromagnetic waves, wavelength, frequency, period*) An electromagnetic wave has a wavelength of $5.0 \times 10^{-10}\,\text{m}$. Determine

(a) the frequency,

(b) the period,

(c) the angular frequency

of the wave.

Q10 (*interference and diffraction*) A plane wave of monochromatic light of wavelength λ is perpendicularly incident on an opaque screen containing two narrow slits separated by a distance $d < \lambda$. The light passes through the slits and is detected on a distant screen placed parallel to the opaque screen. A two-slit interference pattern is observed with a number of peaks and troughs in intensity. Decide whether each of the following statements is true or false.

(a) At the central point on the detecting screen, equidistant from both slits, there is a maximum in intensity.

(b) The distance between successive peaks and troughs of the interference pattern increases as the slit separation d is increased.

(c) The distance between successive peaks and troughs of the interference pattern increases as the wavelength λ is increased.

4 Mathematical skills

Mathematics is a vital tool in quantum physics — it provides the language in which ideas are expressed and gives methods that allow quantitative conclusions to be drawn. You will therefore need to be fluent with algebraic manipulation, vectors, differentiation and integration. In addition, quantum physics makes use of some special mathematical techniques including: vectors, matrices, differential equations, partial differentiation and partial differential equations. All these topics are covered in the prerequisite mathematics module MST224 or MST210 or are developed as part of SM358.

The following 19 questions test your understanding of some of the mathematical techniques that we assume you will be able to use at the outset of your studies of SM358. You should be able to complete these questions in about two hours.

Q11 (*roots of a quadratic equation*) Find the solutions of the quadratic equation $3x^2 - x - 1 = 0$. (Give your answer in terms of $\sqrt{13}$.)

Q12 (*complex numbers, modulus and complex conjugate*) Given $z = 3 - 2\text{i}$ (where $\text{i} = \sqrt{-1}$) write down

(a) $\text{Im}(z)$, the imaginary part of z,

(b) the complex conjugate z^* (or $\overline{z}$) of z,

(c) z^2,

(d) $|z|^2$.

Q13 (*complex numbers, Euler's formula*) Given $z = \mathrm{e}^{\mathrm{i}kx}$ where k and x are real, determine

(a) $\mathrm{Re}(z)$, the real part of z,

(b) $\frac{1}{2}(z + z^*)$,

(c) $|z|$.

Q14 (*summation symbol*) Evaluate

(a) $\displaystyle\sum_{j=1}^{3} j$

(b) $\displaystyle\sum_{n=1}^{4} (n^2 + 1)$.

Q15 (*vectors*) (a) Find the scalar product (or dot product) of the two vectors $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{b} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(b) Show that the two vectors $\mathbf{a}$ and $\mathbf{b}$ point in orthogonal directions.

Q16 (*vectors*) Find the unit vector in the direction of the vector $\mathbf{a}$ in Q15. What is the component of the vector $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in the direction of $\mathbf{a}$?

Q17 (*matrices, transpose of a matrix, matrix multiplication*) Given

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},$$

determine

(a) $\mathbf{Ca}$, (b) $\mathbf{CD}$, (c) $\mathbf{a}^{\mathrm{T}}\mathbf{b}$.

Q18 (*eigenvalues and eigenvectors of a matrix*)

Is $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ an eigenvector of $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$?

If so, what is the corresponding eigenvalue?

Q19 (*differentiation*) Differentiate the following with respect to x.

(a) $y = x^2 \sin x$,

(b) $y = \dfrac{\sin x}{x^2}$,

(c) $y = 5\mathrm{e}^{-x^2}$,

(d) $y = \log_{\mathrm{e}}\left(1 + x^2\right)$.

Q20 (*finding extrema*) Find and characterize the local extremum of the function $f(x) = x\mathrm{e}^{-2x}$.

Q21 (*linear differential equations*)

(a) Find the values of k for which $\cos(kt)$ and $\sin(kt)$ are solutions of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 4y = 0.$$

(b) Write down an arbitrary linear combination of the solutions found in part (a). What property of the differential equation guarantees that this linear combination is a solution?

Q22 (*initial conditions and particular solutions of a differential equation*) Given that your linear combination in part (b) of Q21 is the general solution of the differential equation in part (a), determine the particular solution with the initial conditions

$$y(0) = y'(0) = 1.$$

Q23 (*changing the variable of integration*) Evaluate

(a) $\displaystyle\int_0^{\pi/k} \sin(kx)\,\mathrm{d}x$,

(b) $\displaystyle\int_{-\infty}^{\infty} x^4 \mathrm{e}^{-x^2/a^2}\,\mathrm{d}x$ for $a > 0$,

given that $\displaystyle\int_{-\infty}^{\infty} x^4 \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{3\sqrt{\pi}}{4}$.

Q24 (*integration of even and odd functions*) Given that $u(x)$ is an even function and $v(x)$ is an odd function, determine

(a) $\displaystyle\int_{-Q}^{Q} v(x)\,\mathrm{d}x$,

(b) $\displaystyle\int_{-P}^{P} x u(x)\,\mathrm{d}x$.

Q25 (*partial differentiation*) If $f(x, y) = x^3 y + y$, determine $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y^2$.

Q26 (*evaluating partial derivatives*) Given $f(x, t) = \cos(\pi t - x)$, evaluate $\partial f/\partial t$ at $(x = 0, t = 1)$ and $\partial f/\partial x$ at $(x = \pi, t = \frac{1}{2})$.

Q27 (*plane polar coordinates*) Express the potential energy function $V(x, y) = \frac{1}{2}(Ax^2 + By^2)$ in plane polar coordinates. For what value of B is the force a central force, i.e. one which depends only on the distance from the origin?

Q28 (*multiple integrals*)

Evaluate $\displaystyle\int_{y=0}^{y=2} \int_{x=0}^{x=1} x^2 y\,\mathrm{d}x\,\mathrm{d}y$.

Q29 (*volume integrals and spherical polar coordinates*)

Evaluate the volume integral $I = \int_B r^2\,\mathrm{d}V$ using spherical polar coordinates, where B is a spherical volume of radius a, centred on the origin.

5 Other skills

You should have the following skills

Study skills The ability to:

- organize time for study, and pace study appropriately;
- read effectively and identify relevant information;
- seek help when it is required.

Writing skills The ability to:

- write coherently;
- give succinct and complete definitions;
- write a scientific account with appropriate equations and diagrams.

Problem-solving and modelling skills The ability to:

- recognize the physical principles and equations that apply in described situations;
- translate a problem described in words into a form suitable for mathematical analysis;
- recognize information supplied implicitly or explicitly;
- draw appropriate diagrams;
- check answers and interpret a mathematical solution in physical terms.

ICT skills The ability to:

- use applications and simulation software;
- access information on the Web;
- communicate using email and conferencing software.

6 Suggested further reading and preparatory work

If the questions show that you need to learn more about certain topics or improve certain skills, the following S217 units are recommended:

Units 1–10
Unit 14
Unit 17
Unit 18
Units 23–26

Many other physics textbooks describe the key physics concepts needed for SM358. We recommend:

Fundamentals of physics by D. Halliday, R. Resnick and J. Walker

Physics by H. Ohanian

Physics for Scientists and Engineers by P. Tipler

Quantum Mechanics by F. Mandl

Mathematical skills

If you need to refresh your knowledge of complex numbers or differentiation and integration, Unit 1 of MST224 is recommended.

The following MST224 Units are also especially relevant for SM358:

Units 2 and 3 on differential equations
Unit 7 on partial differentiation
Unit 8 on multiple integrals
Unit 12 on partial differential equations

Two extensive mathematical texts at an appropriate level to prepare for SM358 are *Basic Mathematics for the Physical Sciences* and *Further Mathematics for the Physical Sciences*, both by R. Lambourne and M. Tinker.

7 Solutions to questions

Q1 (a) The momentum is $\mathbf{p} = m\mathbf{v}$ so the magnitude of momentum is

$$\begin{aligned} p &= mv \\ &= 3.0\times10^{-6}\,\mathrm{kg}\times5.0\times10^{-2}\,\mathrm{m\,s^{-1}} \\ &= 1.5\times10^{-7}\,\mathrm{kg\,m\,s^{-1}}. \end{aligned}$$

(b) The kinetic energy is

$$\begin{aligned} E_{\mathrm{kin}} &= \tfrac{1}{2}mv^2 \\ &= \tfrac{1}{2}\times3.0\times10^{-6}\,\mathrm{kg}\times\left(5.0\times10^{-2}\,\mathrm{m\,s^{-1}}\right)^2 \\ &= 3.75\times10^{-9}\,\mathrm{J} \\ &= 3.8\times10^{-9}\ \mathrm{J} \end{aligned}$$

(to two significant figures).

Q2 The kinetic energy $E_{\mathrm{kin}} = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}$. The momentum is $\mathbf{p} = m\mathbf{v}$, so

$$E_{\mathrm{kin}} = \frac{\mathbf{p}\cdot\mathbf{p}}{2m} = \frac{p_x^2+p_y^2+p_z^2}{2m}.$$

Q3 Since $V(x)$ is a function of x only, the force acts only in the x-direction. Use $F_x = -\mathrm{d}V/\mathrm{d}x = -4Cx^3$. With $C = 5.0\,\mathrm{J\,m^{-4}}$, the force at $x = 2.0\,\mathrm{m}$ is
$F_x = -4\times5.0\,\mathrm{J\,m^{-4}}\times(2.0\,\mathrm{m})^3 = -1.6\times10^2\,\mathrm{N}$.

Q4 (a) Using the conservation of energy,

$$E = E_{\text{kin}} + V,$$

so the kinetic energy is

$$E_{\text{kin}} = E - V = -2\,\text{J} - (-5\,\text{J}) = 3\,\text{J}.$$

(b) $E = E_{\text{kin}} + V$. Therefore

$$E_{\text{kin}} = -2\,\text{J} - (-2\,\text{J}) = 0\,\text{J}.$$

(c) E_{kin} must be positive or zero, so $3\,\text{m} \leqslant x \leqslant 12\,\text{m}$.

(d) The only non-zero component of the force is $F_x = -\text{d}V/\text{d}x$. At $x = 11\,\text{m}$, the slope $\text{d}V/\text{d}x$ is positive, so $F_x = -\text{d}V/\text{d}x < 0$. The force is directed in the negative x-direction, i.e. towards the minimum of the potential energy function.

Q5 (a) The angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times (m\mathbf{v}) = m\mathbf{r} \times \mathbf{v}.$$

(b) The vector product of any two parallel or antiparallel vectors is equal to the zero vector, so $\mathbf{L} = \mathbf{0}$.

(c) In this case we have

$$\mathbf{L} = \mathbf{r} \times m\mathbf{v} = 9.0\,\text{kg}\,\text{m}^2\,\text{s}^{-1}\,\mathbf{j} \times \mathbf{i} = -9.0\,\text{kg}\,\text{m}^2\,\text{s}^{-1}\,\mathbf{k}.$$

Q6 (a) Applying Newton's second law of motion, $F_x = ma_x$, gives

$$-kx = m\frac{\text{d}^2x}{\text{d}t^2} \quad \text{or} \quad \frac{\text{d}^2x}{\text{d}t^2} + \frac{k}{m}x = 0.$$

(b) The equation of motion in part (a) describes simple harmonic motion with angular frequency $\omega = (k/m)^{1/2}$.

(c) The kinetic energy of the particle is $E_{\text{kin}} = \frac{1}{2}mv_x^2$ and the potential energy is $V = \frac{1}{2}kx^2$, so the total energy is $E = \frac{1}{2}mv_x^2 + \frac{1}{2}kx^2$.

Q7 (a) The magnitude of the Coulomb force is

$$\begin{aligned} F &= \frac{1}{4\pi\varepsilon_0}\left|\frac{q_1q_2}{r^2}\right| \\ &= 8.99 \times 10^9\,\text{N}\,\text{m}^2\,\text{C}^{-2} \times \frac{\left(1.60 \times 10^{-19}\,\text{C}\right)^2}{\left(1.00 \times 10^{-11}\,\text{m}\right)^2} \\ &= 2.30 \times 10^{-6}\,\text{N}. \end{aligned}$$

(b) The potential energy is

$$\begin{aligned} V &= \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r} \\ &= 8.99 \times 10^9\,\text{N}\,\text{m}^2\,\text{C}^{-2} \times \frac{\left(1.60 \times 10^{-19}\,\text{C}\right)^2}{1.00 \times 10^{-11}\,\text{m}} \\ &= 2.30 \times 10^{-17}\,\text{J}. \end{aligned}$$

Q8 (a) The fixed ends of the string are points of zero displacement, so a whole number of half-wavelengths must fit into the length L of the string. Hence the wavelengths of standing waves obey $n\lambda/2 = L$, where n is a positive integer.

(b) A point of constant phase has $kx + \omega t = \text{constant}$ so $k\text{d}x/\text{d}t + \omega = 0$ and $\text{d}x/\text{d}t = -\omega/k$. The speed of the wave is ω/k and it moves in the negative x-direction.

Q9 (a) Frequency:

$$f = \frac{c}{\lambda} = \frac{3.00 \times 10^8\,\text{m}\,\text{s}^{-1}}{5.0 \times 10^{-10}\,\text{m}} = 6.0 \times 10^{17}\,\text{Hz}.$$

(b) Period:

$$T = \frac{1}{f} = \left(6.0 \times 10^{17}\,\text{Hz}\right)^{-1} = 1.7 \times 10^{-18}\,\text{s}.$$

(c) Angular frequency:

$$\begin{aligned} \omega &= 2\pi f = 2\pi \times 6.0 \times 10^{17}\,\text{Hz} \\ &= 3.8 \times 10^{18}\,\text{s}^{-1}. \end{aligned}$$

Q10 (a) True; waves arriving from the two slits are in phase at the central point, so there is constructive interference leading to a maximum in intensity.

(b) False; the distance between successive peaks and troughs of the interference pattern *decreases* as the slit separation d increases. For example, the first minimum occurs when waves from one slit are half a wavelength out of phase with waves from the other slit. If the slits are moved further apart, this condition is satisfied for a smaller angle of deflection relative to the incident beam, leading to a shorter distance between successive peaks and troughs of the interference pattern.

(c) True; when the wavelength λ is increased, the condition for the first minimum requires a larger angle of deflection relative to the incident beam, leading to a longer distance between successive peaks and troughs of the interference pattern.

Q11 Using the quadratic equation formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 3$, $b = -1$ and $c = -1$, we obtain

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 3 \times (-1)}}{2 \times 3} \\ &= \frac{1 \pm \sqrt{1 + 12}}{6} \\ &= \frac{1 \pm \sqrt{13}}{6}. \end{aligned}$$

Q12 Given $z = 3 - 2\mathrm{i}$,

(a) $\mathrm{Im}(z) = -2$.

(b) $z^* = 3 + 2\mathrm{i}$.

(c)
$$\begin{aligned} z^2 &= (3-2\mathrm{i})^2 \\ &= (3-2\mathrm{i})(3-2\mathrm{i}) \\ &= 9 - 6\mathrm{i} - 6\mathrm{i} + (-2\mathrm{i})^2 \\ &= 9 - 12\mathrm{i} + (-4) \\ &= 5 - 12\mathrm{i}. \end{aligned}$$

(d)
$$\begin{aligned} |z|^2 &= z^* z = (3+2\mathrm{i})(3-2\mathrm{i}) \\ &= 9 + 6\mathrm{i} - 6\mathrm{i} + 4 \\ &= 13. \end{aligned}$$

Q13 Given $z = \mathrm{e}^{\mathrm{i}kx}$,

(a) $z = \cos(kx) + \mathrm{i}\sin(kx)$, so $\mathrm{Re}(z) = \cos(kx)$.

(b)
$$\begin{aligned} \tfrac{1}{2}(z + z^*) &= \tfrac{1}{2}\left[(\cos(kx) + \mathrm{i}\sin(kx)) + (\cos(kx) - \mathrm{i}\sin(kx))\right] \\ &= \cos(kx). \end{aligned}$$

(c) $|z|^2 = z^* z = \mathrm{e}^{-\mathrm{i}kx} \times \mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{-\mathrm{i}kx + \mathrm{i}kx} = \mathrm{e}^0 = 1$. Since $|z| \geqslant 0$, we have $|z| = +1$.

Q14 (a) $\displaystyle\sum_{j=1}^{3} j = 1 + 2 + 3 = 6.$

(b)
$$\begin{aligned} \sum_{n=1}^{4}(n^2+1) &= (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) \\ &= 2 + 5 + 10 + 17 = 34. \end{aligned}$$

Q15 (a) The scalar product of the two vectors is
$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= (2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\cdot(3\mathbf{i} + 2\mathbf{j} + \mathbf{k}) \\ &= 2\times 3 + (-2)\times 2 + (-2)\times 1 \\ &= 6 - 4 - 2 = 0. \end{aligned}$$

(b) The scalar product $\mathbf{a}\cdot\mathbf{b} = ab\cos\theta$, where $a = |\mathbf{a}|$, $b = |\mathbf{b}|$ and θ is the angle between the directions of the two vectors. The vectors $\mathbf{a}$ and $\mathbf{b}$ have non-zero magnitudes so $a \neq 0$ and $b \neq 0$ and the equation $ab\cos\theta = 0$ requires that $\cos\theta = 0$. This implies that the two vectors point in orthogonal directions.

Q16 The magnitude of the vector $\mathbf{a}$ is
$$|\mathbf{a}| = \sqrt{2^2 + (-2)^2 + (-2)^2} = \sqrt{12}$$
so the unit vector in the direction of $\mathbf{a}$ is
$$\widehat{\mathbf{a}} = \frac{1}{\sqrt{12}}(2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}).$$

The component of $\mathbf{c} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ in the direction of $\mathbf{a}$ is
$$\begin{aligned} \widehat{\mathbf{a}}\cdot\mathbf{c} &= \frac{1}{\sqrt{12}}(2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})\cdot(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) \\ &= \frac{1}{\sqrt{12}}\left(2\times 1 + (-2)\times(-2) + (-2)\times 1\right) \\ &= \frac{1}{\sqrt{12}}(2 + 4 - 2) = \frac{2}{\sqrt{3}}. \end{aligned}$$

Q17 (a) $\mathbf{Ca} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

(b) $\mathbf{CD} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}.$

(c) $\mathbf{a}^{\mathrm{T}}\mathbf{b} = [1 \quad 0]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 + 0 = -1.$

Q18 We have
$$\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix} = -2\begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

So $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ is an eigenvector of $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$ with eigenvalue -2.

Q19

(a) Using the product rule with $y = x^2\sin x$, we have
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x^2\frac{\mathrm{d}}{\mathrm{d}x}\sin x + \frac{\mathrm{d}x^2}{\mathrm{d}x}\sin x \\ &= x^2\cos x + 2x\sin x. \end{aligned}$$

(b) Using the quotient rule with $y = \sin x/x^2$, we have
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^2\frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x\frac{\mathrm{d}}{\mathrm{d}x}x^2}{x^4} \\ &= \frac{x^2\cos x - 2x\sin x}{x^4} \\ &= \frac{x\cos x - 2\sin x}{x^3}. \end{aligned}$$

(c) Using the chain rule with $y = 5\mathrm{e}^{-x^2}$, we have
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 5\mathrm{e}^{-x^2}\times\frac{\mathrm{d}}{\mathrm{d}x}\left(-x^2\right) \\ &= -10x\,\mathrm{e}^{-x^2}. \end{aligned}$$

(d) Using the chain rule with $y = \log_{\mathrm{e}}(1 + x^2)$, we have
$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1+x^2}\times\frac{\mathrm{d}}{\mathrm{d}x}(1+x^2) \\ &= \frac{2x}{1+x^2}. \end{aligned}$$

Q20 Differentiating $f(x) = x\mathrm{e}^{-2x}$ once gives

$$f'(x) = \mathrm{e}^{-2x} - 2x\mathrm{e}^{-2x} = (1 - 2x)\mathrm{e}^{-2x}.$$

At an extremum, $f'(x) = 0$. So the extremum occurs at $x = 1/2$, and $f(1/2) = (1/2)\mathrm{e}^{-1}$ at the extremum. To find out whether this extremum is a minimum, maximum or point of inflection we use the second-derivative test. The second derivative of the function is

$$\begin{aligned} f''(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(1 - 2x)\mathrm{e}^{-2x} \\ &= (-2)\mathrm{e}^{-2x} - 2(1 - 2x)\mathrm{e}^{-2x} = -4(1 - x)\mathrm{e}^{-2x}. \end{aligned}$$

This is negative at $x = 1/2$, so the extremum is a *maximum*.

Q21 (a) For $y = \cos(kt)$,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -k\sin(kt) \quad \text{and} \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -k^2\cos(kt).$$

Substituting into the differential equation gives $k^2 = 4$, so $k = \pm 2$. Similarly for $\sin(kt)$ we find $k = \pm 2$.

(b) The solutions in part (a) are $y = \cos(2t)$ and $y = \pm\sin(2t)$. An arbitrary linear combination of them is $y = \alpha\cos(2t) + \beta\sin(2t)$ where α and β are arbitrary constants. The fact that the differential equation is *linear* guarantees that this linear combination is a solution.

Q22 Given

$$y(t) = \alpha\cos(2t) + \beta\sin(2t),$$

we have

$$y'(t) = -2\alpha\sin(2t) + 2\beta\cos(2t).$$

The first initial condition ($y(0) = 1$) gives $1 = \alpha$ while the second initial condition ($y'(0) = 1$) gives $1 = 2\beta$. Hence $\alpha = 1$ and $\beta = 1/2$ and the particular solution is

$$y = \cos(2t) + \tfrac{1}{2}\sin(2t).$$

Q23 (a) Put $u = kx$, then $x = u/k$ and $\mathrm{d}x = (1/k)\,\mathrm{d}u$. The limits of integration $x = 0$ and $x = \pi/k$ correspond to $u = 0$ and $u = \pi$. Hence

$$\begin{aligned} \int_{x=0}^{x=\pi/k} \sin(kx)\,\mathrm{d}x &= \frac{1}{k}\int_{u=0}^{u=\pi} \sin u\,\mathrm{d}u \\ &= \frac{1}{k}\Big[-\cos u\Big]_{u=0}^{u=\pi} \\ &= \frac{2}{k}. \end{aligned}$$

(b) Put $u = x/a$, then $x = au$ and $\mathrm{d}x = a\,\mathrm{d}u$. The limits of integration $x = -\infty$ and $x = \infty$ correspond to $u = -\infty$ and $u = \infty$. Hence

$$\begin{aligned} \int_{x=-\infty}^{x=\infty} x^4\mathrm{e}^{-x^2/a^2}\,\mathrm{d}x &= a^5\int_{u=-\infty}^{u=\infty} u^4\mathrm{e}^{-u^2}\,\mathrm{d}u \\ &= \frac{3a^5\sqrt{\pi}}{4}. \end{aligned}$$

Q24 (a) $v(x)$ is odd, so $v(-x) = -v(x)$. Changing the variable of integration from x to $u = -x$ gives

$$\begin{aligned} I &= \int_{x=-Q}^{x=Q} v(x)\,\mathrm{d}x \\ &= \int_{u=Q}^{u=-Q} v(-u)\,(-\mathrm{d}u) \\ &= (-1)^3\int_{u=-Q}^{u=Q} v(u)\,\mathrm{d}u, \end{aligned}$$

where we have obtained two additional minus signs in the last step by reversing the limits of integration and using the fact that $v(-u) = -v(u)$. Whether the variable of integration is written as x or u cannot affect the value of a definite integral, so we conclude that $I = -I$, and hence $I = 0$. The integral of any odd function over a range centred on the origin is always equal to zero.

(b) If $u(x)$ is an even function, $g(x) = xu(x)$ is an odd function because

$$g(-x) = -xu(-x) = -xu(x) = -g(x).$$

Hence, using the final comment in the answer to part (a), $\int_{-P}^{P} xu(x)\,\mathrm{d}x = 0$.

Q25 Given $f(x, y) = x^3y + y$, we have

$$\begin{aligned} \frac{\partial f}{\partial x} &= 3x^2y, \\ \frac{\partial f}{\partial y} &= x^3 + 1, \\ \frac{\partial^2 f}{\partial x^2} &= 6xy, \\ \frac{\partial^2 f}{\partial y^2} &= 0, \\ \frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\big(3x^2y\big) = 3x^2, \\ \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\big(x^3 + 1\big) = 3x^2. \end{aligned}$$

Q26 Given $f(x,t) = \cos(\pi t - x)$, we have

$$\frac{\partial f}{\partial t} = -\pi \sin(\pi t - x) \quad \text{and} \quad \frac{\partial f}{\partial x} = \sin(\pi t - x).$$

Hence

$$\left.\frac{\partial f}{\partial t}\right|_{x=0,\,t=1} = -\pi \sin(\pi - 0) = 0$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\pi,\,t=\frac{1}{2}} = \sin\left(\tfrac{\pi}{2} - \pi\right) = -1.$$

Q27 Use $x = r\cos\phi$ and $y = r\sin\phi$. Then

$$\begin{aligned} V(r,\phi) &= \frac{1}{2}\left(Ar^2\cos^2\phi + Br^2\sin^2\phi\right) \\ &= \frac{r^2}{2}\left(A\cos^2\phi + B\sin^2\phi\right). \end{aligned}$$

The force is central if V depends on r only. This requires $B = A$ giving

$$V(r) = \frac{Ar^2}{2}\left(\cos^2\phi + \sin^2\phi\right) = \frac{Ar^2}{2}.$$

(*Note*: strictly speaking $V(r)$ and $V(x,y)$ are different functions and so different symbols should be used. However, use of the same symbol is a common abuse of notation which you will come across frequently in physics texts, including SM358.)

Q28 (a) The integral is

$$\begin{aligned} \int_{y=0}^{y=2}\int_{x=0}^{x=1} x^2 y \, \mathrm{d}x \, \mathrm{d}y &= \int_{y=0}^{y=2} y \left(\int_{x=0}^{x=1} x^2 \, \mathrm{d}x\right) \mathrm{d}y \\ &= \int_{y=0}^{y=2} y \left[\frac{x^3}{3}\right]_{x=0}^{x=1} \mathrm{d}y \\ &= \int_0^2 \frac{y}{3} \, \mathrm{d}y \\ &= \left[\frac{y^2}{6}\right]_{y=0}^{y=2} \\ &= \frac{2}{3}. \end{aligned}$$

Q29 The integrand does not depend on the spherical polar coordinates θ and ϕ. The quickest way of doing the integral in this case is to split the sphere into many spherical shells. A typical spherical shell has radius r, thickness δr and volume $4\pi r^2\,\delta r$. The volume integral is then

$$\int_B f(r)\,\mathrm{d}V = \int_0^a r^2 \times 4\pi r^2 \,\mathrm{d}r = 4\pi \int_0^a r^4 \,\mathrm{d}r = \frac{4\pi a^5}{5}.$$

Alternatively, we can use the following method (which would work in more general cases):

$$\begin{aligned} \int_B f\,\mathrm{d}V &= \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} r^2 \times r^2 \sin\theta \,\mathrm{d}\phi\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi} 2\pi \times r^4 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}r \\ &= \int_0^a 2 \times 2\pi \times r^4 \mathrm{d}r = \frac{4\pi a^5}{5}. \end{aligned}$$